Lehmer's Problem and the Newton polygon

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Abstract

Lehmer's famous question concerns the existence of monic integer coefficient polynomials with Mahler measure smaller than a certain constant. Despite significant partial progress, the problem has not been fully resolved since its formulation in 1933. A powerful result independently proven by Lawton and Boyd in the 1980s establishes a connection between the classical Mahler measure of single variable polynomials and the generalized Mahler measure of multivariate polynomials. This led to speculation that it may be possible to answer Lehmer's question in the affirmative with a multivariate polynomial although the general consensus among researchers today is that no such polynomial exists. We show that each possible candidate among a particular class of two variable polynomials can be bi-rationally mapped onto a polynomial with Mahler measure greater than Lehmer's constant. Such bi-rational maps are expected to preserve the Mahler measure for large values of a certain parameter.

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1 The Mahler Measure and Lehmer's Problem

In this chapter we give the historical motivation for Lehmer's problem and outline some partial progress that has been made.

1.1 The Mahler Measure and Kronecker's Theorem

Let $p(x)$ be a monic polynomial with integer coefficients.
We write p as

$$p(x) = \prod_{j=1}^{n}(x - \alpha_j).$$

Consider the quantity $\Delta_{p,k}$ defined as

$$\Delta_{p,k} = \prod_{j=1}^{n}(\alpha_j^k - 1).$$

One can show that $\Delta_{p,k}$ is an integer for all k and its prime factors are subject to certain congruence conditions modulo k which results in it being easier to factor than an arbitrary integer. Exploiting these congruence conditions, D.H. Lehmer searched for large primes in the integer sequences $\{\Delta_{p,k}\}_{k\in\mathbb{N}}$. He found that the growth rate of this sequence is depends on a certain function of p, which we define presently.

Definition 1.1. *The exponential Mahler measure of a polynomial $p(x) = a\prod_{j=1}^{n}(x - \alpha_j)$ is the value*

$$M(p) := |a|\prod_{j=1}^{n}\max\{1, |\alpha_j|\}.$$

The logarithmic Mahler measure of p is

$$m(p) := \log M(p).$$

Throughout the following, Mahler measure will be understood to mean exponential Mahler measure.
We list a few elementary properties of m.

Proposition 1.1. *For any polynomials $p(x)$ and $q(x)$ in $\mathbb{Z}[x]$ we have the following:*

(1) $M(pq) = M(p)M(q)$

(2) $M(p) = 1 \iff p$ is a product of a monomial and a cyclotomic polynomial.

(1) is immediate from the definition. (2) follows from a well-known theorem of Kronecker. To prove it, we make use of the following lemma.

Lemma 1.2. *The set of integer coefficient polynomials, of fixed degree n, with all zeroes inside or on the unit circle is finite.*

Proof. Let $p(x) = a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$ be such a polynomial and let $x_1, ..., x_n$ be its zeroes. We have

$$|a_j| = \left|\sum_{\substack{I\subset\{1,...,n\}\\|I|=n-j}}\prod_{k\in I}x_k\right| \leq \binom{n}{j}$$

for each k, $|x_k| \leq 1$. Then we have at most $2\binom{n}{j} + 1$ choices for each coefficient and so set of polynomials satisfying our requirements is finite. $\square$

Recall that an algebraic integer is a zero of a monic polynomial with integer coefficients. The minimal polynomial of an algebraic integer α is the unique monic irreducible polynomial p of smallest degree such that $p(\alpha) = 0$. Two algebraic integers are conjugates of one another if their minimal polynomials coincide.

Theorem 1.3 (Kronecker)**.** *Suppose α is an algebraic integer such that all of its conjugates lie inside or on the unit circle. Then α is a root of unity.*

Proof. Let $p(x) = \prod_{j=1}^{n}(x-\alpha_j)$ be the minimal polynomial of $\alpha = \alpha_1$. We claim the polynomial

$$p_K(x) = \prod_{j=1}^{n}(x-\alpha_j^K)$$

has integer coefficients for any integer K.
Indeed, all coefficients of p_K are a symmetric polynomial evaluated at $(x_1,...,x_n)$ and so are fixed by the Galois group $\mathrm{Gal}(\mathbb{Q}(x_1,...,x_n)/\mathbb{Q})$. Then they must be rational algebraic integers.
By the previous lemma, the set $\{p_K : K \in \mathbb{N}\}$ is finite and so for some $K, L \in \mathbb{N}$, we have $p_K = p_L$ and so we can assume W.L.O.G that

$$\alpha_1^K = \alpha_2^L.$$

Repeatedly applying this reasoning, and by appropriately labelling the zeroes of p, we see that for some integer $1 \leq a \leq n$ we have $\alpha_j^K = \alpha_{j+1}^L$, for $1 \leq j \leq a$, where subscripts are taken modulo a.
Then

$$\alpha_1^{K^a} = \alpha_1^{L^a}$$

and so α_1 is an $(K^a - L^a)$th root of unity.

□

1.2 Lehmer's Problem

For Lehmer's task, it would be helpful to use polynomials with small Mahler measure greater than 1, for the corresponding sequences $\{\Delta_{p,k}\}_{k\in\mathbb{N}}$ would grow slowly. Lehmer found the polynomial

$$l(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$$

which has Mahler measure $\lambda = 1.17628018....$ λ is the smallest known constant, greater than 1, which is the Mahler measure of a monic integer polynomial. The problem of determining a monic integer polynomial p such that

$$1 < m(p) < \lambda$$

is known as Lehmer's problem. Many believe no such polynomial exists and that λ is an optimal lower bound.

On inspection of Lehmer's polynomial, we observe that its coefficients are palindromic.
Polynomials with such coefficients are said to be reciprocal. Reciprocity of a polynomial $p(x)$ of degree n can be characterised algebraically by the equation

$$p(x) = x^n p(x^{-1}).$$

As the following result of C.J. Smyth shows, reciprocity is a very important property in the context of Lehmer's question.

Theorem 1.4 (Smyth)**.** *Let $p(x)$ be a monic polynomial with integer coefficients. If p is not reciprocal*

$$M(p) \geq \theta_0$$

where $\theta_0 = M(x^3 - x - 1) > \lambda$.

A version of this result with a smaller lower bound was proved earlier by Breusch. [7]
We now define a generalised Mahler measure.

Definition 1.2. *Let $p(x_1,...,x_n)$ be a Laurent polynomial with complex coefficients. The logarithmic Mahler measure of p is the value*

$$m(p) := \int_0^1 ... \int_0^1 \log|p(e^{2\pi i\theta_1},...,e^{2\pi i\theta_n})| d\theta_1...d\theta_n$$

The exponential Mahler measure of p is

$$M(p) := \exp m(p).$$

The following theorem, known as Jensen's formula, shows that the two definitions coincide in the single variable case.

Theorem 1.5 (Jensen)**.** *For any* $\alpha \in \mathbb{C}$ *we have*

$$\int_0^1 \log|e^{2\pi it} - \alpha|dt = \log^+|\alpha|$$

where $\log^+|\alpha| = \max\{0, \log|\alpha|\}$.

Proof. We write α in polar as $\alpha = re^{2\pi is}$, $r > 0$, $s \in [0,1]$.
The integral under consideration becomes

$$\begin{aligned}\int_0^1 \log|e^{2\pi it} - \alpha|dt &= \int_0^1 \log|e^{2\pi it} - re^{2\pi is}|dt \\ &= \int_0^1 \log|e^{2\pi i(t-s)} - r|dt + \int_0^1 \log|e^{2\pi is}|dt \\ &= \int_0^1 \log|e^{2\pi it} - r|dt\end{aligned}$$

For now suppose $r \neq 1$ We now write this last integral as a limit of approximating sums by partitioning the unit circle into n sectors of equal area.
This gives

$$\begin{aligned}\int_0^1 \log|e^{2\pi it} - r|dt &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \log|e^{2\pi ik/n} - r| \\ &= \lim_{n\to\infty} \frac{1}{n} \log|\prod_{k=1}^{n}(e^{2\pi ik/n} - r)| \\ &= \lim_{n\to\infty} \log|1 - r^n|^{1/n}\end{aligned}$$

Since

$$\lim_{n\to\infty} |1 - r^n|^{1/n} = \begin{cases} 1, 0 < r < 1 \\ r, r > 1 \end{cases}$$

we obtain the result.
If $r = 1$ the integral is improper. Let γ_ϵ be the circular arc of radius $\epsilon > 0$ and centre 1 such that its endpoints, z_1, z_2 lie on the unit circle and the rest of the curve lies inside. Let Γ_ϵ be unit circle with the shorter arc connecting z_1 and z_2 removed. We parametrize γ_ϵ from z_1 to z_2 and Γ_ϵ from z_2 to z_1.
Then

$$\begin{aligned}\int_0^1 \log|e^{2\pi it} - r|dt &= \lim_{\epsilon\to 0} \frac{1}{2\pi i} \int_{\Gamma_\epsilon} \log|z-1|\frac{dz}{z} \\ &= \lim_{\epsilon\to 0} \frac{1}{2\pi i} \int_{\gamma_\epsilon} \log|z-1|\frac{dz}{z}\end{aligned}$$

where the last equality follows from Cauchy's theorem. The integral in the last expression has size at most $\epsilon|\log\epsilon|$ which tends to 0 as ϵ tends to 0. □

Jensen's formula shows that the two definitions coincide when p is a polynomial in a single variable but it is not immediately clear that general formula is well defined. That is, if $p(x_1, .., x_n)$ vanishes on the torus $\mathbb{T}^n$, then this point is a singularity of $\log|p(e^{2\pi i\theta_1}, ..., e^{2\pi i\theta_n})|$ and so it is not clear that the integral converges. It turns out that not only is the integral defining the Mahler measure guaranteed to converge, it can be realised as the limit of Mahler measures of polynomials in a single variable via the following technical result.

Theorem 1.6 (Lawton, Boyd)**.** *Let* $r = (r_1, ..., r_n) \in \mathbb{Z}^n$ *and let* $q(r) := \min\{H(s) : s = (s_1, ..., s_n) \in \mathbb{Z}^n, s \neq (0, ..., 0), \sum_{j=1}^{n} s_j r_j = 0\}$ *where* $H(s) := \max\{|s_j| : 1 \leq j \leq n\}$. *For a Laurent polynomial* $p(x_1, ..., x_n)$ *let* $p_r(x) = p(x^{r_1}, ..., x^{r_n})$.
Then

$$M(p) = \lim_{q(r)\to\infty} M(p_r).$$

In particular, in the case of a Laurent polynomial in two variables, we have

$$M(p(x,y)) = \lim_{n\to\infty} M(p(x,x^n)).$$

This result expands Lehmer's problem to multivariate polynomials. If there exists a monic polynomial $p(x,y)$, with integer coefficients, such that $M(p(x,y)) < \lambda$, then for some n, $M(p(x,x^n)) < \lambda$.

2 Convex Lattice Polygons

A convex lattice polygon is the convex hull of a finite number of points in the integer lattice $\mathbb{Z}^2$.
the acronym CLP will be used for convex lattice polygon. One can place a natural equivalence relation on the set of CLPs. A result of P.R. Scott implies there are finitely many equivalence classes for a fixed number of interior lattice points. In this section, we classify all equivalence classes of convex lattice triangles with one interior lattice point. The proofs used here appear in a paper of Rabinowitz [8] where all CLPs with a single interior point are characterised. Throughout this section $\mathcal{P}$ will denote a convex lattice polygon. $A(\mathcal{P})$, $b(\mathcal{P})$ and $i(\mathcal{P})$ will denote the area, number of boundary lattice points and number of interior lattice points of $\mathcal{P}$, respectively. Given a line segment AB with integer endpoints, the lattice length of AB is one less than the number of integer points on AB.

2.1 Pick's Formula and Scott's Upper Bound on Boundary Points

Definition 2.1. *Two convex lattice polygons $\mathcal{P}$ and $\mathcal{Q}$ are said to be lattice equivalent if there exists $M \in SL_2(\mathbb{Z})$ and $v \in \mathbb{Z}^2$ such that*

$$\mathcal{P} = M(\mathcal{Q}) - v.$$

Mappings of the form $x \mapsto Mx - v$ are known as an affine unimodular transformations.
One observes that the quantities A, b and i are invariant under equivalence.
In our study of CLPs, it will be useful map line segments onto the x-axis. The following lemma shows this can always be achieved in an affine unimodular fashion.

Lemma 2.1. *Let A and B be lattice points such that the line segment AB has lattice length p. There exists an affine unimodular transformation which sends A to $(0,0)$ and B to $(p,0)$.*

Proof. We first translate A to the origin. Let (x,y) be the image of B under this translation. We observe that the lattice length p is given by

$$p = \gcd(x,y).$$

We require integers a, b, c and d which satisfy $ad - bc = \pm 1$ and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ 0 \end{pmatrix}.$$

Let $c = \frac{-y}{\gcd(x,y)}$ and let $d = \frac{x}{\gcd(x,y)}$. We have $cx + dy = 0$, as required. Using the Euclidean algorithm, we can choose a and b so that

$$ax + by = \gcd(x,y) = p.$$

Then we have

$$\begin{aligned} ad - bc &= a\left(\frac{x}{p}\right) - b\left(\frac{-y}{p}\right) \\ &= \frac{1}{p}(ax + by) \\ &= 1 \end{aligned}$$

□

Definition 2.2. *A shear of weight k about the x-axis is a transformation of the form*

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix},$$

$k \in \mathbb{Z}$.

A shear of weight k about the x-axis sends the point (x,y) to $(x + ky, y)$. We observe that points on the x-axis are fixed by such a transformation. We also note that any shear about the x-axis has unit determinant and thus constitutes an affine unimodular transformation.
In 1899 Pick established a formula for the area of a lattice polygon, which need not be convex, that depends only on the number of boundary and interior lattice points.

Theorem 2.2 (Pick)**.**

$$A(\mathcal{P}) = i(\mathcal{P}) + \frac{1}{2}b(\mathcal{P}) - 1$$

As we will see, this is a useful tool when arguing about CLPs. Unfortunately the formula does not generalize to higher dimensional polytopes.
In 1976 Scott proved the following, appealing to Pick's formula:

Theorem 2.3 (Scott)**.**

$$b(\mathcal{P}) \leq 2i(\mathcal{P}) + 7$$

Proof. Our task is to demonstrate the inequality

$$f(\mathcal{P}) := b(\mathcal{P}) - 2i(\mathcal{P}) \leq 7.$$

Pick's formula gives the following equivalent definitions of f:

$$f(\mathcal{P}) = b(\mathcal{P}) - A(\mathcal{P}) - 1,$$

$$f(\mathcal{P}) = A(\mathcal{P}) - 2i(\mathcal{P}) + 1.$$

We say $y = cx + d$ is a supporting line of $\mathcal{P}$ if for all $(i, j) \in \mathcal{P}$, $i \leq aj + b$ or for all $(i, j) \in \mathcal{P}$, $i \geq aj + b$. We can assume that $\mathcal{P}$ lies in the non-negative quadrant and meets supporting lines $y = 0$ and $y = l$ for some $l \in \mathbb{N}$. Suppose $\mathcal{P}$ meets $y = l$ in a segment of length h and meets $y = 0$ in a segment of length k.
By the convexity of $\mathcal{P}$, each horizontal line between $y = 0$ and $y = l$ can meet $\mathcal{P}$ in at most 2 points and so we obtain

$$b(\mathcal{P}) \leq h + k + 2l.$$

l must be at least 2 for otherwise $\mathcal{P}$ would have no interior point.
Consider the case $l = 2$, $h + k \geq 4$ or $l = h + k = 3$. Again appealing to the convexity of $\mathcal{P}$, we see that $\mathcal{P}$ must contain a trapezium with bases of length h and k and height l.
Then

$$A(\mathcal{P}) \geq \frac{1}{2}l(h + k)$$

whence

$$\begin{aligned} f(\mathcal{P}) &= 2b(\mathcal{P}) - 2A(\mathcal{P}) - 2 \\ &\leq 2(h + k + 2l) - l(h + k) - 2 \\ &= (h + k - 4)(2 - l) + 6 \\ &\leq 7. \end{aligned}$$

We now consider the case $l = 3$, $h + k \leq 2$.
We have

$$b(\mathcal{P}) \leq h + k + 2l \leq 8.$$

Then since $\mathcal{P}$ has at least one interior point,

$$f(\mathcal{P}) = b(\mathcal{P}) - 2i(\mathcal{P}) \leq 6.$$

The final case where $l \geq 4$ and $h + k \leq 3$ relies on a technical lemma not proven here. For details, we direct the reader to [6] □

2.2 Polygons with No Interior Lattice Points

Definition 2.3. *$\mathcal{T}_p^h$ will be used to denote the triage with vertices $(0, 0)$, $(p, 0)$ and $(0, h)$. $\mathcal{Q}_{p,q}^h$ will be used to denote the quadrilateral with vertices $(0, 0)$, $(p, 0)$, $(0, h)$ and (q, h).*

We first classify convex lattice triangles with no interior lattice points.

Theorem 2.4. *Let $\mathcal{P}$ be a convex lattice triangle with $i(\mathcal{P}) = 0$. Then $\mathcal{P}$ is lattice equivalent to either $\mathcal{T}_2^2$ or $\mathcal{T}_p^1$, for some positive integer p.*

Proof. We label the vertices of $\mathcal{P}$, A, B and C. We assume AC is the side with greatest lattice length. Transform $\mathcal{P}$ so that A is mapped to $(0,0)$, C is mapped to $(p,0)$, and B is mapped to a point above the x-axis. This is possible by Lemma 2.1. Let h be the height of B above the x-axis. If $h=1$, B is of the form $(k,1)$. We can apply a shear of weight $-k$ about the x-axis. This map fixes A and C and sends B to the $(0,1)$. This shows $\mathcal{P}$ is equivalent to $\mathcal{T}_p^1$.
Then we assume $h \geq 1$.
Let r be the length of the line segment obtained by intersecting the line $y=1$ with $\mathcal{P}$ and let D and E be the endpoints of this line segment. Then ABC and DBE are similar triangles and so we obtain

$$\frac{r}{h-1} = \frac{p}{h}$$

which gives

$$r = \frac{p(h-1)}{h}.$$

We observe that r cannot be greater than 1 since then $\mathcal{P}$ would contain an interior lattice point and so we obtain

$$\frac{p(h-1)}{h} \leq 1.$$

One way this inequality can be satisfied is if $p=1$ in which case $\mathcal{P}$ would be equivalent to $\mathcal{T}_1^1$. Then we must have $p \geq 2$.
Rearranging the above inequality we have

$$\begin{aligned} \frac{h}{h-1} \geq p &\iff 1 + \frac{1}{h-1} \geq p \\ &\iff \frac{1}{h-1} \geq p-1 \\ &\iff h \leq 1 + \frac{1}{p-1} \end{aligned}$$

We already know h must be at least 2 and this last inequality shows that it can be at most 2. Hence $h=2$ which forces $p=2$. Since EF has lattice length 1 and contains no lattice points in it's interior, E must be a lattice point. Since $p=2$, the only possibilities for E are $(0,1)$ and $(1,1)$. In the former case $\mathcal{P}$ is precisely $\mathcal{T}_2^2$. If $E=(1,1)$, we apply a shear of weight -1 about the x-axis which transforms $\mathcal{P}$ into $\mathcal{T}_2^2$. □

We now demonstrate that any convex lattice quadrilateral with no two sides parallel must contain and interior lattice point.

Theorem 2.5. *Let $\mathcal{P}$ be a convex lattice quadrilateral with no two sides parallel. Then $i(\mathcal{P}) \geq 1$.*

Proof. Let $\mathcal{P}$ be a convex lattice quadrilateral with vertices A, B, C and D. Suppose that $i(\mathcal{P})=0$ no two sides of $\mathcal{P}$ are parallel and $i(\mathcal{P})=0$. Then the diagonal AC has lattice length 1 and so the triangle ABC is equivalent to $\mathcal{T}_p^1$ for some integer $p \geq 1$ by Theorem 2.4.
We transform $\mathcal{P}$ so that A lies at $(0,0)$, C lies at $(1,0)$ and B lies at $(0,p)$. The situation is shown in Figure 1.

D cannot lie to the left of the line $x=0$ since $\mathcal{P}$ would fail to be convex. D cannot lie on the line $x=0$ since $\mathcal{P}$ would reduce to a triangle. D cannot lie on the line $x=1$ since the segments AB and CD would be parallel. Finally, D cannot lie on or above the line $y=p(1-x)$ since $\mathcal{P}$ would reduce to a triangle or fail to be convex.
We consider the cases $p=1$ and $p \geq 2$ separately.
Suppose $p=1$. D cannot lie below the line $y=-x$ since $(1,-1)$ would be an interior point. However, D must lie below $y=1-x$, as argued above. Then D can only lie on the line $y=-x$ which results in the segments BC and AD being parallel.
Then suppose $p \geq 2$. If AD passes through or above the point $(1,-1)$ then AD will meet the line $x=2$ in the half plane $y \geq -2$. On the other hand, $y=p(1-x)$ meets the line $x=2$ in the half plane $y \leq -2$. That is, the only way AD can pass through or above $(1,-1)$is if D lies on or above $y=p(1-x)$ which is impossible. Then $(1,-1)$ lies above AD and so $(1,-1)$ is in the interior of $\mathcal{P}$, a contradiction. □

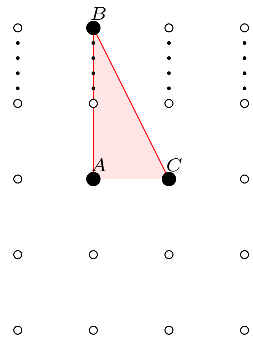

Figure 1

We are now in a position to classify all convex lattice quadrilaterals with no interior lattice points.

Theorem 2.6. *Let $\mathcal{P} = ABCD$ be a convex lattice quadrilateral with $i(P) = 0$. Then $\mathcal{P}$is equivalent to $\mathcal{Q}^1_{p,q}$ for some positive integers $p \geq q \geq 1$.*

Proof. By Theorem 2.5, a pair of edges of $\mathcal{P}$ must be parallel. We assume W.L.O.G that AB is parallel to CD. We assume further that AB has lattice length $p \geq 1$, CD has lattice length $q \geq 1$ and $p \geq q$. Let h be the height of CD above the x-axis.
The segment BC must have lattice length one since $\mathcal{P}$ has no interior points. Then the triangle ABC must be equivalent to $\mathcal{T}^1_p$ by Theorem 2.4.
The area of ABC is clearly

$$A(\mathcal{P}) = \frac{ph}{2}.$$

On the other hand, by Pick's formula we find

$$A(\mathcal{P}) = \frac{p+2}{2} - 1 = \frac{p}{2}$$

and so we deduce that $h = 1$. Applying a shear about the x-axis to map C onto the y-axis, we see that $\mathcal{P}$ is equivalent to $\mathcal{Q}^1_{p,q}$ as required.

□

The classification of convex lattice polygons with $i(\mathcal{P}) = 0$ ends here for, as we now demonstrate, any convex lattice pentagon must contain an interior lattice point.

Theorem 2.7. *Let $\mathcal{P}$ be a convex lattice pentagon. Then $i(\mathcal{P}) \geq 1$.*

Proof. Suppose $\mathcal{P} = ABCDE$ is a convex lattice pentagon with $i(\mathcal{P}) = 0$. Then the quadrilateral $ABCD$ contains no lattice points in it's interior and is therefore equivalent to $\mathcal{Q}^1_{p,q}$ for some positive integers $p \geq q \geq 1$. We assume W.L.O.G that AB is parallel to CD where AB has lattice length p and CD has lattice length q. We can also assume that $A = (0, 0)$, $B = (0, p)$, $C = (1, q)$ and $D = (1, 0)$. The situation is shown in Figure 2. E cannot lie to the left of the line $x = 0$ or to the right of the line $x = 1$ since $\mathcal{P}$ would fail to be convex. Then E must lie on $x = 0$ or $x = -1$ which is a contradiction since $\mathcal{P}$ then reduces to a quadrilateral. □

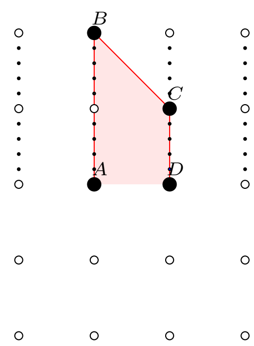

Figure 2

2.3 Triangles with One Interior Lattice Point

We first list all distinct equivalence classes of convex lattice triangles with a single interior point.

Theorem 2.8. *Let $\mathcal{P} = ABC$ be a convex lattice triangle with $i(\mathcal{P}) = 1$. Then $\mathcal{P}$ is equivalent to one of the five triangles depicted in Figure 3.*

Proof. We assume W.L.O.G that AB is the side with greatest lattice length p and that A lies at the origin and B at $(p, 0)$. We also assume C lies above the x-axis. Let h be the height of C above the x-axis. By Pick's formula we have

$$\begin{aligned} A(\mathcal{P}) &= i(\mathcal{P}) + \frac{b(\mathcal{P})}{2} - 1 \\ &= \frac{b(\mathcal{P})}{2} \end{aligned}$$

On the other hand, the area is clearly $\frac{ph}{2}$ and so we have $ph = b(\mathcal{P})$.
By assumption, p is the greatest lattice length of any side of $\mathcal{P}$ and so $b(\mathcal{P}) \leq 3p$.
Then

$$ph \leq 3p$$

whence

$$h \leq 3.$$

If $h = 1$, $\mathcal{P}$ would contain no interior lattice points and so we need only consider the cases $h = 2$ and $h = 3$. As in the proof of Theorem 2.4, let r be the length of the line segment obtained by intersecting the line $y = 1$ with $\mathcal{P}$. We must have $r \leq 2$ or $\mathcal{P}$ would have more than one interior point.
By similar triangles we have

$$\frac{r}{h-1} = \frac{p}{h}.$$

Making use of the bounds $r \leq 2$ and $2 \leq h \leq 3$ we have

$$p = \frac{hr}{h-1} \leq \frac{2h}{h-1} \leq 4.$$

Case 1: $h = 2$.
We have $p = 2r \leq 4$ and so $p = 1, 2, 3$ or 4. Then point C lies on the line $y = 2$. We can assume that $C = (1, 2)$ or $C = (0, 2)$, for if not, we can apply a shear about the x-axis.

Case 1a: $h = 2$, $p = 4$.
Choosing $C = (0, 2)$ yields precisely polygon (a) in Figure 3. C cannot be the point $(1, 2)$ for the resulting polygon has two interior points.

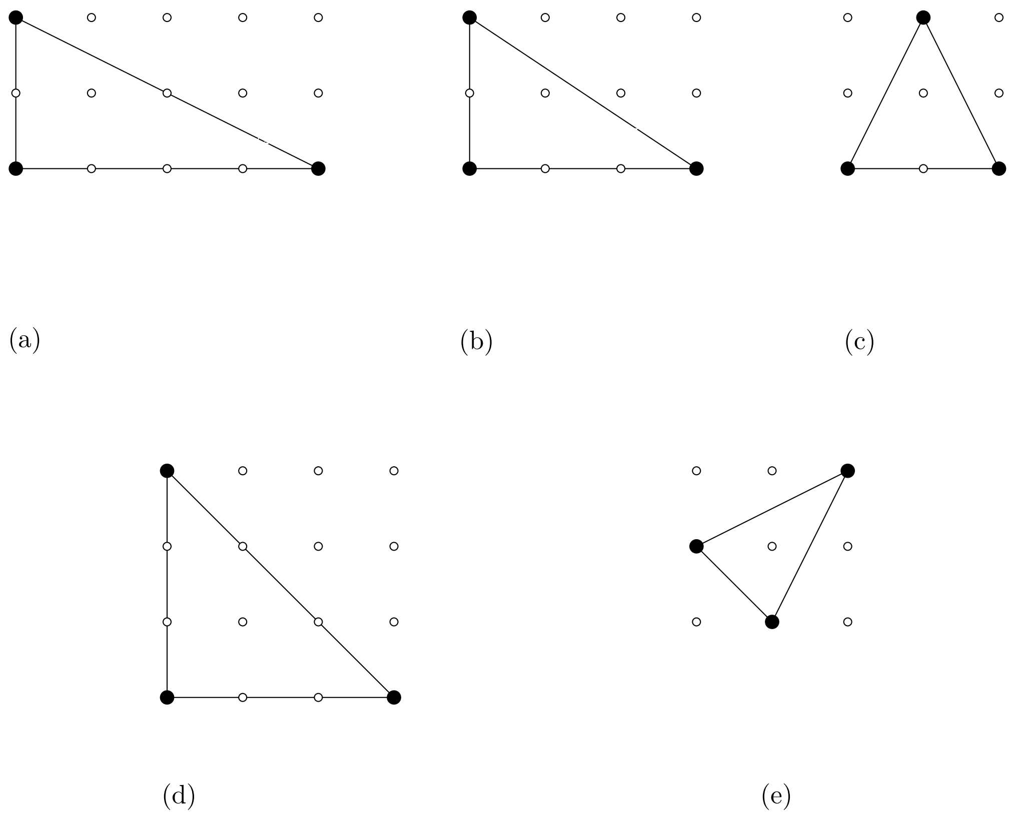

Figure 3: A complete list of inequivalent convex lattice triangles with a single interior point.

Case 1b: $h = 2$, $p = 3$. If we choose $C = (0, 2)$, we obtain polygon (b) in Figure 3. Setting $C = (1, 2)$ yields another triangle with a single interior point but this is also equivalent to polygon (b) if we apply the transformation

$$\begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$$

after first translating the polygons so that the interior point lies at the origin.
Case 1c: $h = 2$, $p = 2$.
If $C = (0, 2)$, we obtain the triangle $\mathcal{T}_2^2$ which has no interior lattice points. Choosing $C = (1, 2)$ yields precisely polygon (c) in Figure 3.

Case 1d: $h = 2$, $p = 1$. Both possibilities for C yield triangles with no interior lattice points.

Case 2: $h = 3$. We have $p = 3r/2 \leq 3$ and so $p = 1, 2$ or 3. C lie on the line $y = 3$. We can assume C lies at $(0, 3)$, $(1, 3)$ or $(2, 3)$, for if not, we can apply an appropriate shear about the x-axis.

Case 2a: $h = 3$, $p = 3$. Both choices $C = (1, 3)$ and $C = (2, 3)$ yield triangles with more than one interior point. Setting $C = (0, 3)$ yields precisely polygon (d) in Figure 3.

Case 2b: $h = 3$, $p = 2$.
The triangle obtained by setting $C = (1, 3)$ contains two interior lattice points. The choice $C = (0, 3)$ yields precisely polygon (b) in Figure 3. The triangle obtained by setting $C = (2, 3)$ is also equivalent to polygon (b) via a shear of weight one about the x-axis.

Case 2c: $h = 3$, $p = 1$.
Setting $C = (0, 3)$ or $C = (1, 3)$ results in triangles with no interior points. The choice $C = (2, 3)$ yields a triangle equivalent to polygon (e). This can be seen by applying the transformation

$$\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$$

after first translating both polygons so the interior point lies at the origin. $\square$

Continuing along these lines of argument one can indeed characterise all CLPs with a single interior lattice point. [8]
There are sixteen unique representatives and we show them below in Figure 4.

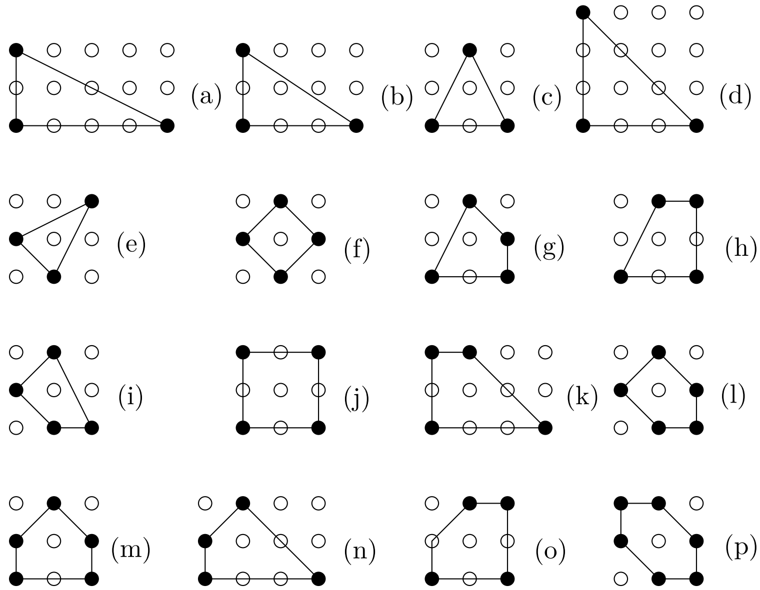

Figure 4

3 The Newton Polygon and Reciprocal Polynomials

3.1 The Newton Polytope

Throughout the following 'polynomial' will be understood to mean Laurent polynomial, unless otherwise indicated. A convex lattice polytope is the convex hull of finitely many points in the integer lattice $\mathbb{Z}^n$. Associated to each polynomial is a certain convex lattice polytope.

Definition 3.1. *The Newton polytope of* $p(x,y) = \sum_{v \in \mathbb{Z}^n} a(v) z_1^{v_1} ... z_n^{v_n}$, $a(v) \in \mathbb{C}$, *is defined as*

$$\mathcal{N}_p := convex.hull\{v : a(v) \neq 0\}.$$

A lattice point $v \in \mathcal{N}_p$ *is labelled with the coefficient* $a(v)$ *(which may be 0). When* $n = 2$, $\mathcal{N}_p$ *is called the Newton Polygon of* p.

Recall that a hyperplane in $\mathbb{R}^n$ is a set of the form

$$H = \{v \in \mathbb{R}^n : \langle w, v \rangle = d\}$$

for some $d \in \mathbb{R}$ and some normal vector $w \in \mathbb{R}^n$. $H = \{v \in \mathbb{R}^n : \langle w, v \rangle = d\}$ is said to be a supporting hyperplane of the polytope K if either $\langle x, w \rangle \geq d$ or $\langle x, w \rangle \leq d$ for all $x \in K$. Finally, $F \subset K$ is a face of the polytope K if K has a supporting hyperplane containing F.

Definition 3.2. *Let* F *be a face of* $\mathcal{N}_p$. *The face polynomial of* L *is defined as*

$$q_F(z_1, ..., z_n) = \sum_{v \in F} a(v) z_1^{v_1} ... z_n^{v_n}$$

where $a(v)$ *is the label of* v *in* $\mathcal{N}_p$.

Let V be an $n \times n$ matrix with integer entries and $\det V = \pm 1$. Then the change of variables

$$(z_1, ..., z_n) \mapsto (z_1, ..., z_n)V \tag{1}$$

fixes the torus $\mathbb{T}^n$ and as a result, it has no affect on the integral defining the Mahler measure. This motivates the following definition.

Definition 3.3. *Two polynomials related by a changes of variables as described above and multiplication by a monomial are said to be equivalent.*

If we identify that the exponents of an n-variable polynomial lie in some hyperplane or lower dimensional affine subspace, we will be able to reduce to a polynomial in fewer variables and thereby simplify the Mahler measure integral.

Example 3.1. *Suppose we wish to compute the Mahler measure of the polynomial*

$$f(x,y) = 1 + x^2y + x^4y^2.$$

Applying the change of variables $(x,y) \mapsto (xy^{-1}, x^{-1}y^2)$*, which is of the form (1), we obtain*

$$f^*(x,y) = 1 + x + x^2$$

which we recognise as cyclotomic and so we can deduce that

$$M(f) = M(f^*) = 1.$$

From this we see that every polynomial in two variables has as its face polynomials, single variable polynomials and so we can make the following equivalent definition.

Definition 3.4. *Let* F *be a face of the Newton polygon of* $p(x,y)$ *and let* $l_0, ..., l_k$ *be all lattice points on* F*, listed in counter-clockwise orientation. The face polynomial of* F *is defined as*

$$q_F(z) = \sum_{j=0}^{k} c_j z^j$$

where c_j *is the label of* l_j *in* $\mathcal{N}_p$*.*

Example 3.2. *The Newton Polygon of the polynomial* $f(x,y) = 1 + x + y$ *is shown in Figure 5.*
It's face polynomials are

$$q_1(z) = q_2(z) = q_3(z) = 1 + z.$$

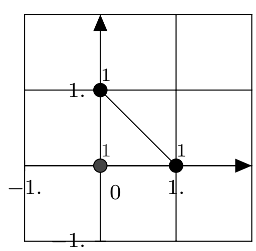

Figure 5

As shown by the following result of Smyth and Boyd, which is the basis of our interest in the Newton polytope, the Mahler measure of a polynomial is bounded below by the greatest of the Mahler measures of it's face polynomials. The proof presented here appears in [3].

Theorem 3.1 (Smyth, Boyd)**.** *Let* $p(z_1, ..., z_n) = \sum_{v \in \mathbb{Z}^n} a(v) z_1^{v_1} ... z_n^{v_n}$ *and let* F *be a face of* $\mathcal{N}_p$*. Then*

$$M(p) \geq M(q_F).$$

Proof. We argue under the assumption that F has codimension 1. The general case then follows by induction. Since F is a face, it is contained in some supporting hyperplane $H = \{x \in \mathbb{Z}^n : \langle x, b \rangle = d\}$. We can choose the normal vector $b = (b_{11}, b_{21} ..., b_{n1})$ to have coprime integer entries. By a classical result of number theory, we can find an integer $n \times n$ matrix B with first column b^T and determinant 1. We define new variables w_j by

$$(z_1, .., z_n) = (w_1, ..., w_n)B$$

and set

$$r(w_1, ..., w_n) = p(z_1, .., z_n).$$

Since each monomial in p is of the form

$$\prod_{i=1}^{n} z_i^{j_i} = \prod_{k=1}^{n} w_k^{\sum_{i=1}^{n} j_i b_{ik}}$$

we have

$$r(w_1, ..., w_n) = \sum_{v \in \mathcal{N}_p B} a(vB^{-1}) w_1^{v_1} ... w_n^{v_n}.$$

The set FB is contained in the hyperplane $HB = \{x \in \mathbb{Z}^n : x_1 = d\}$. We write $r(w_1, ..., w_n)$ as a sum of terms of the form $r_j(w_2, ..., w_n) w_1^j$ where r_j is a Laurent polynomial in $w_2, ..., w_n$. Since HB is a supporting hyperplane, we can assume W.L.O.G that the greatest value of the index j is d, replacing w_1 by w_1^{-1} and multiplying through by a power of w_1 if needed. Let L be the minimum value of j.
Then

$$\begin{aligned} r(w_1, ..., w_n) &= w_1^L (r_d w_1^{d-L} + r_{d-L-1} + ... + r_L) \\ &= w_1^L r_d (w_1^{d-L} + (r_{d-1}/r_d) + ... + (r_L/r_d)) \\ &= w_1^L r_d R \end{aligned}$$

where R is a rational function in $w_1, ..., w_n$.
Now

$$m(p) = m(r) = m(r_d) + m(R).$$

r_d is precisely the image of q_F under the change of variables defined above and so $m(r_d) = m(q_F)$.
Now consider the quantity

$$m(R) = \int_0^1 ... \int_0^1 \log |R(e^{2\pi i\theta_1}, e^{2\pi i\theta_2} ..., e^{2\pi i\theta_n})| d\theta_1 ... d\theta_n.$$

For any fixed values of $(w_2, ..., w_n)$, $R(w_1, w_2, ..., w_n)$ is a polynomial in w_1. It then follows from Jensen's formula that

$$\int_0^1 \log |R(e^{2\pi i\theta_1}, e^{2\pi i\theta_2} ..., e^{2\pi i\theta_n})| d\theta_1 \geq 0$$

for any fixed values of $(\theta_2, ..., \theta_n)$. This shows that $m(R) \geq 0$.
We can then conclude that

$$m(p) \geq m(q_F)$$

and taking exponentials we obtain the statement of the theorem. □

From this result we see that to optimise the lower bound from the face polynomials, we should restrict our attention to polynomials whose face polynomials all have exponential Mahler measure 1.

3.2 Classification of Tempered Reciprocal Families

In this section 'polynomial' refers to a Laurent polynomial in two variables.

Definition 3.5. *A polynomial is said to be tempered if all of its face polynomials have exponential Mahler measure 1.*

We define the genus of a polynomial to be the number of lattice points in the interior of its Newton polygon. This number is related to the genus of the algebraic variety defined by the zero locus of the polynomial, but as we shall not discuss varieties, we make this distinct definition.
We say two polynomials are birationally equivalent if there exist an invertible map $(x, y) \mapsto (F(X, Y), G(X, Y))$, F and G rational functions over $\mathbb{C}$, under which one is the image of the other up to a factor of exponential Mahler measure 1.
A polynomial is said to be reciprocal if

$$p(x, y) = x^n y^m p(x^{-1}, y^{-1})$$

for some $n, m \in \mathbb{Z}$. Reciprocal polynomials can also be characterised as those whose Newton polygon is invariant under a 180 degree rotation followed by a translation by some $(n, m) \in \mathbb{Z}^2$. Of particular interest

to us are polynomials which are tempered and reciprocal.
If we restrict our attention to polynomials of genus 1, we can classify the tempered reciprocal polynomials up to equivalence. Equivalence of polynomials corresponds to unimodular affine equivalence of Newton polygons, as defined in the last chapter.

Only three of the 16 polygons shown at the end of the previous chapter have 180 degree rotational symmetry and are therefore the only polygons which can corresponding to a reciprocal polynomial.

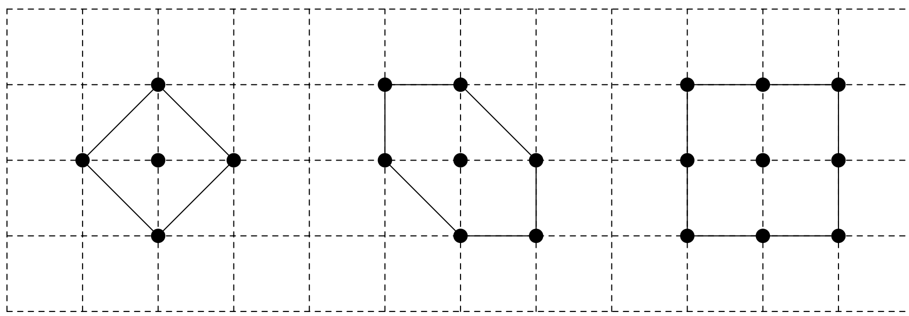

Figure 6

We can determine all tempered reciprocal families, up to a constant factor, by assigning coefficients to the lattice points in these polygons. The coefficient of the interior point, which we take to be the origin, is unrestricted. In two of these polygons, all face polynomials have degree 1 and so the coefficients must be ± 1 to ensure that the polygon is tempered. In the third polygon, the face polynomials have degree 2. The face polynomials are of the form $a + bz + cz^2$ and we can assume a and c are 1 or -1. If $a = c$, b is free to take any value from the set $\{0, \pm 1, \pm 2\}$. If $a \neq c$, the polynomial will fail to be cyclotomic unless $b = 0$. To ensure reciprocity, we also require that each point has the same coefficient as its image under a 180 degree rotation. We summarise this classification in the following theorem.

Theorem 3.2. *An integer coefficient, tempered, reciprocal polynomial of genus 1 is equivalent to one of the following:*

$$\alpha x + \beta y + \frac{\alpha}{x} + \frac{\beta}{y} + k$$

$$\alpha x + \beta y + \frac{\delta y}{x} + \frac{\alpha}{x} + \frac{\beta}{y} + \frac{\delta x}{y} + k$$

$$\alpha xy + \frac{\beta y}{x} + \eta y + \zeta x + \frac{\alpha}{xy} + \frac{\beta x}{y} + \frac{\eta}{y} + \frac{\zeta}{x} + k$$

where $\alpha, \beta, \delta \in \{1, -1\}$, $\eta, \zeta \in \begin{cases} \{0\}, \alpha \neq \beta \\ \{0, \pm 1, \pm 2\}, \alpha = \beta \end{cases}$, $k \in \mathbb{Z}$.

3.3 Birational Maps

Birational maps which preserve the temperedness property are expected to preserve the Mahler measure, at least for sufficiently large values of the internal parameter k. In light of Smyth's result, non reciprocal polynomials are not likely to yield single variable polynomials of small Mahler measure. Then by demonstrating that all tempered, reciprocal polynomials in two variables can be birationally mapped to tempered non reciprocal polynomials, one adds weight to the hypothesis that Lehmer's problem is unsolvable
M.J. Bertin gives the following example. [9]
The image of the polynomial $p(x, y) = y^2 + y(x^2 + kx + 1) + x^2$ under the transformation $(x, y) \mapsto \left(\frac{Y}{X-1}, \frac{kY-(X-1)^2}{(X-1)^2}\right)$ is

$$\frac{-kY}{(X-1)^4}(X^3 - kXY - 2X^2 - Y^2 + X).$$

$y^2+y(x^2+kx+1)+x^2$ is equivalent to $x+y+\frac{1}{x}+\frac{1}{y}+k$ and it is easy to verify that $X^3-kXY-2X^2-Y^2+X$ is tempered and non-reciprocal.
Its Newton polygon is shown below:

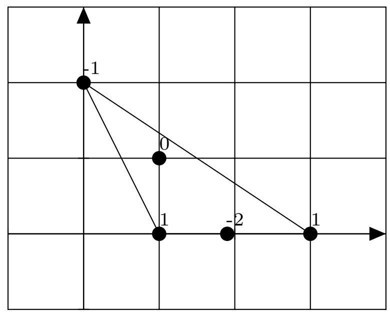

Figure 7

The face polynomials of the three faces are

$$q_1(z) = 1 - 2z + z^2 = (z-1)^2,$$

$$q_2(z) = 1 + z,$$

$$q_3(z) = 1 + z.$$

All three of the face polynomials are cyclotomic and the polygon has no 180 degree rotational symmetry. An important feature of this transformation is that it is invertible and so it definies a birational equivalence. This is why the genus is preserved. We give an example of a transformation which preserves temperedness, breaks reciprocity and does not preserve genus for genus 1 polynomials although this is not expected to affect the Mahler measure for large values of the parameter k.

Before giving the transformation, we outline the intuition which led to its discovery.
Suppose we wish to determine a change of variables which preserves temperedness but breaks the reciprocity of the polynomial $x + y + \frac{1}{x} + \frac{1}{y} + k$.
For simplicity, let us consider transformations of the form

$$(x, y) \mapsto (P, Q)$$

where P and Q are polynomials in X and Y.
Making the substitution and factoring out $\frac{1}{P}$ and $\frac{1}{Q}$, we obtain

$$\frac{1}{PQ}(P^2Q + Q^2P + P + Q + kPQ).$$

If the transformation is to satisfy our requirements in this form, P and Q should have Mahler measure 1.
To break reciprocity, it seems that it should be enough to choose P and Q so they are not symmetric in X and Y. Since k is arbitrary, we can only have a tempered polynomial if all terms with a coefficient depending on k correspond to a point in the interior of the Newton polygon. This seems likely to be the case if the exponents of the terms in PQ are approximately the average of the exponents of all terms in the polynomial.
If we choose P to be of positive degree and Q of negative degree, the degree of kPQ should be between that of P^2Q and Q^2P.
With these requirements in mind, we try the transformation

$$(x, y) \mapsto (X^2Y + Y, X^{-1}Y^{-1}).$$

In this case, $\frac{1}{PQ} = \frac{X}{(X^2+1)}$ and $M\left(\frac{X}{(X^2+1)}\right) = 1$.

As we will see in the proof of the following theorem, this transformation has the desired properties when applied to any tempered reciprocal genus 1 polynomial.

Theorem 3.3. *Given a tempered, reciprocal polynomial of genus 1, one can obtain a tempered non reciprocal polynomial via a birational change of variables.*

Proof. Throughout the following proof, the coefficients $\alpha, \beta, \delta, \eta, \zeta$ are as defined in Theorem 3.2.

The image of the polynomial $p_1(x,y) = \alpha x + \beta y + \alpha\frac{1}{x} + \beta\frac{1}{y} + k$ under the transformation $(x,y) \mapsto ((X^2+1)Y, X^{-1}Y^{-1})$ is

$$\frac{X}{(X^2+1)}(\beta X^2 Y + \alpha Y X^3 + \beta Y + \alpha Y X^{-1} + 2\alpha XY + k(X + X^{-1}) + \beta Y^{-1} + \beta X^{-2}Y^{-1} + \alpha X^{-1}Y^{-1}).$$

Shown below in Figure 8 is the Newton polygon of the polynomial

$$P_1(X,Y) = \beta X^2 Y + \alpha Y X^3 + \beta Y + \alpha Y X^{-1} + 2\alpha XY + k(X + X^{-1}) + \beta Y^{-1} + \beta X^{-2}Y^{-1} + \alpha X^{-1}Y^{-1}.$$

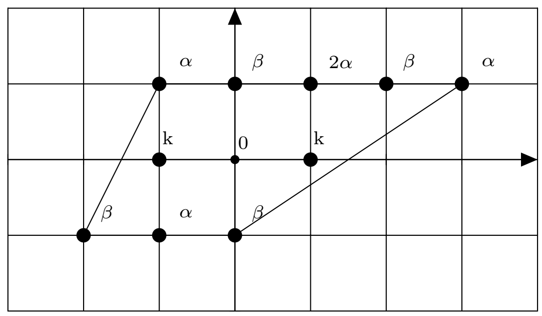

Figure 8

The face polynomials of the four faces are

$$q_1(z) = \alpha + \beta z + 2\alpha z^2 + \beta z^3 + \alpha z^4,$$

$$q_2(z) = \alpha + \beta z,$$
$$q_3(z) = \beta + \alpha z + \beta z^2,$$
$$q_4(z) = \beta + \alpha z.$$

We see that all face polynomials are cyclotomic for all legal choices of coefficients so the polynomial corresponding to this Newton polygon is tempered.
$P_1(X,Y)$ is not reciprocal. To see this, observe that if we rotate its Newton polygon by 180 degrees, we cannot recover the original polygon by a translation.

We now consider the image of $p_2(x,y) = \alpha x + \beta y + \alpha\frac{1}{x} + \beta\frac{1}{y} + \delta\frac{x}{y} + \delta\frac{y}{x} + k$ under $(x,y) \mapsto ((X^2+1)Y, X^{-1}Y^{-1})$.
We obtain

$$\frac{X}{(X^2+1)}(\beta X^2 Y + \alpha Y X^3 + \beta Y + \alpha Y X^{-1} + 2\alpha XY + k(X + X^{-1}) + \beta Y^{-1} + \beta X^{-2}Y^{-1} + \alpha X^{-1}Y^{-1} + \delta X^{-2}Y^{-2} + \delta Y^2 X^4 + 2\delta X^2 Y^2 + \delta Y^2).$$

Below in Figure 9 is the Newton polygon of the polynomial

$$P_2(X,Y) = \beta X^2Y + \alpha YX^3 + \beta Y + \alpha YX^{-1} + 2\alpha XY + k(X + X^{-1}) +$$
$$\beta Y^{-1} + \beta X^{-2}Y^{-1} + \alpha X^{-1}Y^{-1} + \delta X^{-2}Y^{-2} + \delta Y^2X^4 + 2\delta X^2Y^2 + \delta Y^2.$$

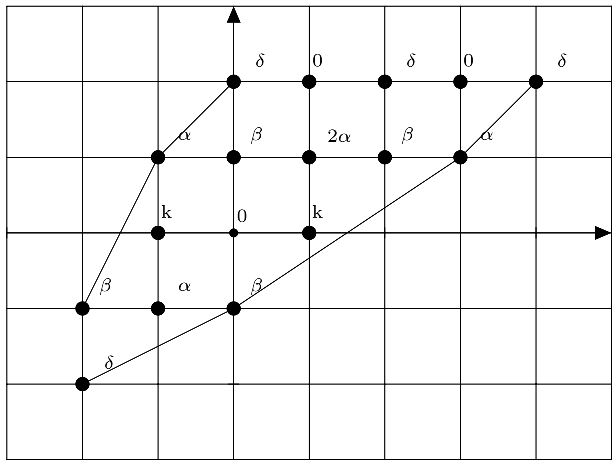

Figure 9

The face polynomials are listed below

$$q_1(z) = \delta + \delta z^2 + \delta z^4,$$
$$q_2(z) = \delta + \alpha z,$$
$$q_3(z) = \alpha + \beta z,$$
$$q_4(z) = \beta + \delta z,$$
$$q_5(z) = \delta + \beta z,$$
$$q_6(z) = \beta + \alpha z,$$
$$q_7(z) = \alpha + \delta z.$$

Again, all of these polynomials are cyclotomic for all legal coefficients, and so $P_2(X,Y)$ is tempered. By the same argument used in the previous case, $P_2(X,Y)$ is not reciprocal.
Now we consider the image of $p_3(x,y) = \alpha xy + \frac{\beta y}{x} + \eta y + \zeta x + \frac{\alpha}{xy} + \frac{\beta x}{y} + \frac{\eta}{y} + \frac{\zeta}{x} + k$ under $(x,y) \mapsto ((X^2+1)Y, X^{-1}Y^{-1})$.
We have

$$\frac{X}{(X^2+1)}(\alpha X^2 + \alpha X^{-2} + \beta X^{-2}Y^{-2} + \beta X^4Y^2 + 2\beta X^2Y^2 + \beta Y^2 +$$
$$\eta Y^{-1} + \eta X^{-2}Y^{-1} + \zeta X^3Y + 2\zeta XY + \zeta X^{-1}Y + \eta X^2Y + \eta Y\zeta X^{-1}Y^{-1} + k(X + X^{-1}) + 3\alpha).$$

Below in Figure 10 is the Newton polygon of

$$P_3(X,Y) = \alpha X^2 + \alpha X^{-2} + \beta X^{-2}Y^{-2} + \beta X^4Y^2 + 2\beta X^2Y^2 + \beta Y^2 +$$
$$\eta Y^{-1} + \eta X^{-2}Y^{-1} + \zeta X^3Y + 2\zeta XY + \zeta X^{-1}Y + \eta X^2Y + \eta Y\zeta X^{-1}Y^{-1} + k(X + X^{-1}) + 3\alpha.$$

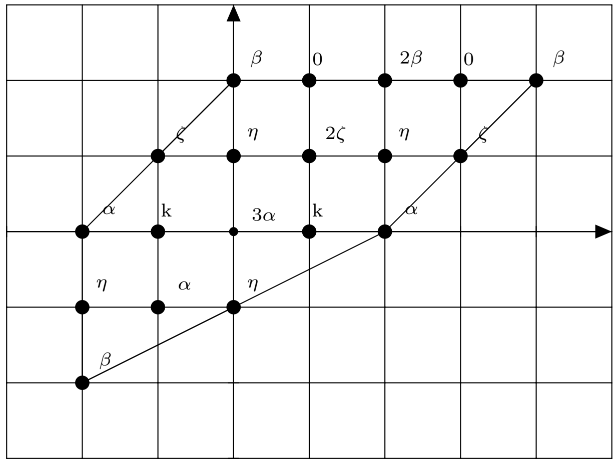

Figure 10

The face polynomials are listed below

$$
\begin{aligned}
q_1(z) &= \beta + 2\beta z^2 + \beta z^4, \\
q_2(z) &= \beta + \zeta z + \alpha z^2, \\
q_3(z) &= \alpha + \eta z + \beta z^2, \\
q_4(z) &= \beta + \eta z + \alpha z^2, \\
q_5(z) &= \alpha + \zeta z + \beta z^2.
\end{aligned}
$$

Once again, all face polynomials are cyclotomic for all legal coefficients and by inspection of the Newton Polygon, we see that $P_3(X,Y)$ is not reciprocal.

$\square$

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